PHYSICAL REVIEW E **70**, 045202(R) (2004)

First-order synchronization transition in locally coupled maps

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(Received 10 February 2004; published 27 October 2004)

We study several diffusively coupled chaotic maps on periodic d-dimensional square lattices. Even and odd sublattices are updated alternately, introducing an effective *delay*. As the coupling strength is increased, the system undergoes a first-order phase transition from a multistable to a synchronized phase. At the transition point, the largest Lyapunov exponent of the system changes sign contrary to the earlier studies which predicted the same to be negative. Further increase in coupling strength shows desynchronization where the phase space splits into two ergodic regions. We argue that the nature of desynchronization transition strongly depends on the differentiability of the maps.

DOI: 10.1103/PhysRevE.70.045202 PACS number(s): 05.45.Xt, 05.45.Ra

Synchronization is observed in a wide class of complex systems. Typically, it appears when the range of the correlations inside the system is of the same order as the system size. Few examples of mutual synchronization in complex dynamical systems are flashing fireflies, electronic circuits, and chemical reactions [1]. In recent years, synchronization of spatially extended systems has drawn considerable interest. Particularly, coupled map lattices (CMLs) [2], initially introduced as simple models of spatio-temporal chaos, have received a great deal of attention as a model of synchronization [3]. It has been realized that two identical chaotic systems, coupled directly [4], or through a common source of external random noise [5,6], can achieve complete synchronization for strong enough coupling strengths. CMLs with global coupling can also achieve mutual synchronization [7]. In all these examples, the transition occurs from an unsynchronized phase in which two replicas evolve independently, to a synchronized phase in which memory of the initial difference is asymptotically lost and then both the systems follow the same chaotic trajectory.

In the synchronized phase the largest Lyapunov exponent Λ of the dynamical system has to be negative [8]. Then depending on whether Λ changes sign at the transition point or before, two different scenarios are possible [6]. In the later case, Λ is negative at the transition point and thus any local fluctuation which desynchronizes some lattice sites would rapidly be reabsorbed due to lack of any mechanism of information propagation. Like the directed percolation (DP) process, in this case, nonsynchronized clusters, can never emerge from already synchronized regions. The synchronization transition (ST) is thus expected to be in the universality class of DP. On the contrary, if Λ changes sign at the transition point, sufficiently close to ST, local fluctuations would eventually grow to generate nonsynchronized clusters, even within the synchronized regions. The growth is, however, bounded from above by the *boundedness* of the primitive maps. Recently a Langevin equation was introduced by Pikovsky and Kurths [9] to model the growth, the saturation, and the noise, which is generalized later by Munõz and Pastor-Satorras [10]. In this model, the natural order parameter of synchronization $\phi(x,t)$, defined as the average of absolute difference between trajectories, satisfies

$$\partial_t \phi = -(a\phi + b\phi^2 + c\phi^3) + D\nabla^2\phi + \eta\phi, \qquad (1)$$

where $D>0$ is the diffusion constant, $\eta(x,t)$ is a Gaussian white noise, and a,b, and c are parameters to model the growth and the saturation. For $b>0$, c is irrelevant, and using a Hopf-Cole transformation, $h=\ln|\phi|$, one can identity Eq. (1) as the Kardar-Parisi-Zhang (KPZ) model [11] with additional saturation terms. Thus the ST is in the universality class of "bounded KPZ" (bKPZ) model.

Note that in the bKPZ language $b<0$ case of Eq. (1) corresponds to the presence of an attractive upper wall. In this case one can argue that Λ is negative at the transition point, and thus the ST is in the DP class. However, a careful analysis [10] reveals that for certain choices of parameters, one can have a highly attractive wall and in this regime, phase transition can occur discontinuously. In the context of nonequilibrium wetting process, such first-order phase transitions (FOPTs) have been observed in several $(1+1)$-dimensional stochastic models with local interactions [12], contrary to equilibrium wetting processes where phase transitions are generically not possible [13] in one-dimensional (1D) systems that have short-range interactions between interfaces and substrates. However, to the best of our knowledge, first-order STs in chaotic, extended systems with short range interactions are still lacking, although they are known to exist for globally coupled maps [14]. In this Rapid Communication we propose to find such a transition.

We mainly study a single parameter family of chaotic piecewise linear maps (PLMs) which are diffusively coupled on a d-dimensional square lattice. An effective *delay* is introduced dynamically between sublattices by updating them alternately. One of our interests would be to find if, starting from a random initial condition, these sublattices synchronize at later times. The answer turns out to be "no," for both very high and low diffusion strengths ϵ. However, for intermediate ϵ, synchronization occurs with the suppression of spatio-temporal chaos. This synchronized phase is a *unique absorbing state* of the system and for PLMs the phase boundary is identical with the stability boundary of the common fixed point of the CML. The desynchronization also occurs discontinuously for strong enough couplings as the common fixed point loses stability. To this end we will discuss the generalizations of the delay-induced STs to other chaotic maps.

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Two main results of this paper can be summarized as follows. First, an effective *delay* introduced between sublattices of CMLs can generically enforce synchronization by suppressing chaos. Second, at the transition point the largest Lyapunov exponent Λ of the CML changes sign, contrary to Eq. (1), which predicts Λ to be negative at the transition point of a first-order transition.

The model. Consider a d-dimensional hypercubic lattice $\mathcal{L}$ of coupled identical maps $f(z_{\vec{i}})$, where $z_{\vec{i}}$ is a real variable at site $\vec{i}\equiv(i_1,i_2,\ldots,i_d)$ with i_k varying from 1 to L. We define the *even* and *odd* sublattices ($\mathcal{L}^e$ and $\mathcal{L}^o$, respectively) as $\mathcal{L}^{e,o}=\{\vec{i}:\Sigma\, i_k=even,odd\}$, and denote $x_{\vec{i}}$ ($y_{\vec{i}}$) as the variable of $\mathcal{L}^e$ ($\mathcal{L}^o$). Starting from a random initial configuration, $\{x_{\vec{i}}\}$ and $\{y_{\vec{i}}\}$ are updated alternately as

$$x_{\vec{i}}^{t+1} = (1-\epsilon)f(x_{\vec{i}}^t) + \frac{\epsilon}{2d}\sum_{\vec{j}\in\mathcal{N}_{\vec{i}}} f(y_{\vec{j}}^t),$$

$$y_{\vec{i}}^{t+1} = (1-\epsilon)f(y_{\vec{i}}^t) + \frac{\epsilon}{2d}\sum_{\vec{j}\in\mathcal{N}_{\vec{i}}} f(x_{\vec{j}}^{t+1}), \tag{2}$$

where $\mathcal{N}_{\vec{i}}$ is a set of d-dimensional nearest neighbors of $\vec{i}$, and $\epsilon>0$ is the coupling strength, can be seen as a diffusion constant. Equivalently, in the first half unit of time $x_{\vec{i}}$ are updated while keeping $y_{\vec{i}}$ unaltered and in the second half only $y_{\vec{i}}$ are updated. We will see later that this *delay*, which is introduced dynamically between sublattices, is responsible for a complete synchronization of the CML. Note that periodic boundary configuration in all d dimensions are used throughout.

Synchronization occurs when the difference between $x_{\vec{i}}$ and $y_{\vec{i}}$ vanishes at all sites as $t\to\infty$. Thus, the order parameter of ST can be defined as $\phi=\langle\phi^t\rangle$ where

$$\phi^t = \frac{1}{L^d}\sum_{\vec{i}\in L^e} |x_{\vec{i}}^t - y_{\vec{i}}^t|, \tag{3}$$

and the steady state average is taken over time and realizations. Obviously, ϕ vanishes in the synchronized phase and in the unsynchronized phase $\phi>0$. A trivial synchronized phase would correspond to the stable fixed point of the CML, i.e., $\{z_{\vec{i}}=z^*\}$. For chaotic CML without delay, this state is linearly unstable. To see this, let us take the Fourier transform of the small deviations $\delta z_{\vec{i}}=z_{\vec{i}}-z^*$. Then, the Fourier coefficients $\delta z_{\vec{k}}$ evolve as $\delta z_{\vec{k}}^{t+1}=E_{\vec{k}}\delta z_{\vec{k}}^t$, with $E_k=\mu(1-\epsilon)+R_{\vec{k}}$. Here, we define

$$\mu = f'(z^*) \quad \text{and} \quad R_{\vec{k}} = \frac{\epsilon\mu}{2d}\sum_{\vec{r}\in\mathcal{N}_0} e^{i\vec{k}\cdot\vec{r}}.$$

Since the primitive maps are chaotic, $|\mu|>1$ and thus $\max(|E_{\vec{k}}|)>1$, which proves that a common fixed point $\{z_{\vec{i}}=z^*\}$ is unstable. The delay introduced in Eq. (2), however, can stabilize the common fixed point in a region $\epsilon_B\leqslant\epsilon\leqslant\epsilon_A$. In this case, the Fourier coefficients of the small deviations $\delta x_{\vec{i}}=x_{\vec{i}}-z^*$ and $\delta y_{\vec{i}}=y_{\vec{i}}-z^*$ evolve as

$$\begin{pmatrix}\delta x_{\vec{k}}\\ \delta y_{\vec{k}}\end{pmatrix}^{t+1} = \begin{pmatrix}\tilde{\mu} & R_{\vec{k}}\\ \tilde{\mu}R_{\vec{k}} & \tilde{\mu}+R_{\vec{k}}^2\end{pmatrix}\begin{pmatrix}\delta x_{\vec{k}}\\ \delta y_{\vec{k}}\end{pmatrix}^{t}, \tag{4}$$

where $\tilde{\mu}=\mu(1-\epsilon)$. Let $E_{\vec{k}}^{\pm}$ denote the eigenvalues of the matrix defined in Eq. (4). From the stability requirements $|\max\{\mathrm{Re}(E_{\vec{k}}^+),\mathrm{Re}(E_{\vec{k}}^-)\}|<1$, we find that, $\epsilon_A=1+1/\mu$ and $\epsilon_B=(\mu-1)/(2\mu)$ which are drawn in the Fig. 1 as a phase boundary for the synchronized phase. For simplicity, it is assumed here that primitive maps have only one nonzero fixed point z^*. One can further generalize it to maps with more fixed points.

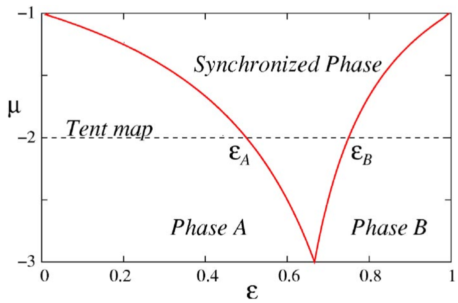

FIG. 1. Phase diagram in ϵ-μ plane for the piece-wise linear maps defined in Eq. (5).

To find out the behavior of ϕ close to these transitions we first restrict ourselves to one dimension and study a specific single parameter family of maps:

$$f(z) = \begin{cases} mz/(m-1), & z<1-1/m\\ m(1-z), & z\geqslant 1-1/m \end{cases}. \tag{5}$$

This piecewise linear mapping of [0, 1] onto itself is everywhere expanding for $m>1$, and thus chaotic, with an invariant density uniform on [0, 1]. A particular example of this family with $m=2$ is known as *tent map*. Note that the fixed point is $z^*=m/(m+1)$.

Synchronization. Let us first discuss the transition from the unsynchronized phase A (see Fig. 1) to the synchronized phase. Close to the transition point we take $\epsilon=\epsilon_A-\delta$ and find that the system becomes multistable as $\delta\to 0$, i.e., one out of the large number of steady states is chosen by the CML, depending on the initial configuration. The delay introduced here could be the possible source of the multistability. Delayed differential equations [15] and CMLs [16] are known to exhibit such behavior. It may be argued that the multistability is extensive, i.e., the number of attractors grow exponentially with the system size. Thus any statistical average has to be taken over a large number of independent realizations, which restricts us to simulate large systems. We carried out numerical simulations for $L=1024$ and $m=2.0$, 1.5, and find that ϕ vanishes discontinuously at $\epsilon_A=1-1/m$ (see Fig. 3). To confirm that it is a true first-order transition, not just a transient effect, we monitor the phase space of every neighboring pair of coordinates as $\delta\to 0$. For example in Fig. 2 we demonstrate how the phase space changes in the z_1-z_2 plane. For large δ, the phase space is identical for two different initial configurations S_1 and S_2. However, dynamically different shapes are generated as $\delta\to 0$. In practice, no noticeable change is observed in the phase space when $\delta<10^{-4}$ and then suddenly the fixed point $z_1=z^*=z_2$ appears at $\delta=0$. Every other pair of neighboring coordinates show similar changes. In other words, when $\delta\approx 0$, we have $|z_i-z_{i+1}|>0$ for every realization and thus ϕ has a jump at $\epsilon=\epsilon_A$.

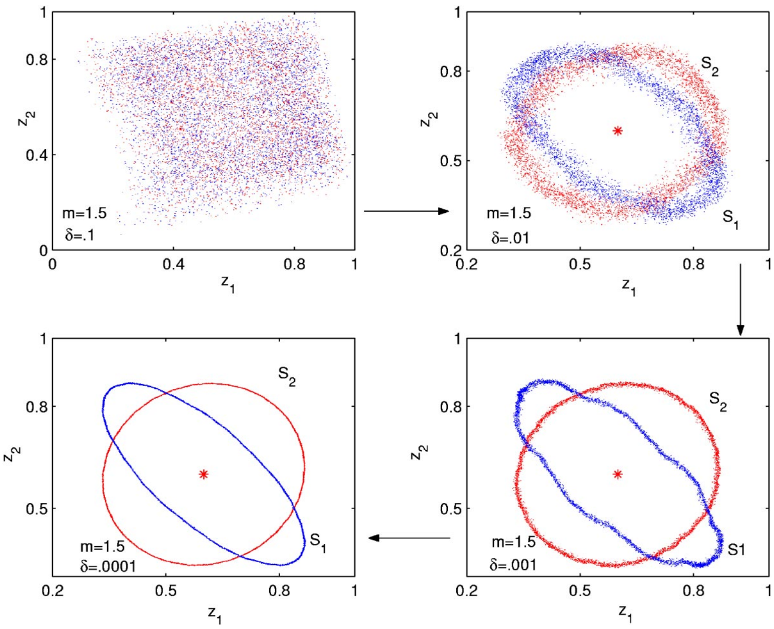

FIG. 2. This figure shows how the phase space changes in z_1-z_2 plane for two different initial configurations S_1 and S_2 as $\delta = \epsilon_A - \epsilon \to 0$. The symbol "*" represents the fixed point $z^* = 0.6$.

From Eq. (1), Λ is expected to be negative at the transition point of a first-order ST. To check this, we evaluate Λ numerically using the standard methods [17]. The results for the map (5) with $m = 1.5$ and 2.0 are illustrated in Fig. 3. Clearly, Λ changes sign exactly at ϵ_A, which suggests that the Langevin equation (1) needs further modification to incorporate the first order STs within the bKPZ scenario ($b > 0$).

Desynchronization. The synchronized state persists until $\epsilon = \epsilon_B$, where phase space splits into *two* disconnected ergodic regions. In the new phase B, shown in Fig. 1, x_i and y_i fluctuate about two different fixed points x^* and y^*, corresponding to even and odd sublattices, respectively. From Eq. (2) we have

$$x^* = (1-\epsilon) f(x^*) + \epsilon f(y^*),$$
$$y^* = (1-\epsilon) f(y^*) + \epsilon f(x^*), \tag{6}$$

which can be solved for the maps defined in (5) as $x^* = \alpha_\pm$ and corresponding $y^* = \alpha_\mp$, where

$$\alpha_\pm = \frac{m^2(2\delta + 1) \pm (2\delta m + 1)}{m^2(2\delta + 1) \pm m(4\delta + 1)}$$

and $\delta = \epsilon - \epsilon_B$. Thus, depending on the initial configuration, different parts within a sublattice can either be attracted to α_+ or α_- with kinklike interfaces separating them. The corresponding counterpart of the other sublattice is then attracted to α_- or α_+ respectively (see Fig. 4). It will be shown later that the width of such a kink w, diverges as $1/\sqrt{\delta}$ as $\delta \to 0$. Thus, stable kinks cannot be generated when δ is $\mathcal{O}(L^{-2})$ and we have $\phi = \alpha_+ - \alpha_-$. Clearly the jump-in ϕ at ϵ_B is

$$\Delta = \lim_{L\to\infty} \lim_{\delta\to 0} (\phi_+ - \phi_-) = 2/(m^2 + m). \tag{7}$$

On the other hand, for a fixed $\delta \approx 0$, thermodynamically large systems would generate different number of kinks for different initial conditions. Taking the average density of kinks to be ρ, one can obtain the jump in ϕ at ϵ_B as $\tilde{\Delta} = \Delta(1 - \rho w) + \rho \mathcal{A}$, where $\mathcal{A}$ is the area bounded by an even and an odd kink (the shaded area shown in Fig. 4). Since $w \sim 1/\sqrt{\delta}$, $\tilde{\Delta} \neq 0$ and thus the desynchronization transition is discontinuous.

To see that $w \sim 1/\sqrt{\delta}$, let us first calculate the steady state profile of a kink which starts at site k with, say $z_k = \alpha_-$ and $z_{k+1} = \alpha_+$. Using the steady state condition $z_k^{t+1} = z_k^t$ in Eq. (2), we find that z_{k+2} has two solutions: α_- and

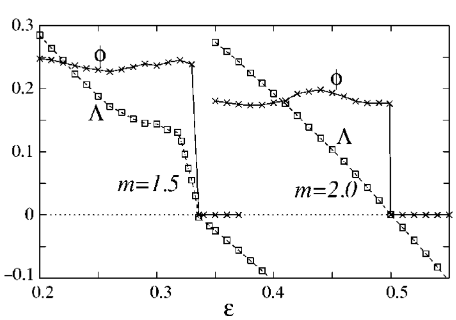

FIG. 3. The largest Lyapunov exponents Λ (shown as squares) are estimated numerically for map (5) with $m = 1.5$ and 2.0. Clearly Λ changes sign at ϵ_A, where ϕ vanishes discontinuously.

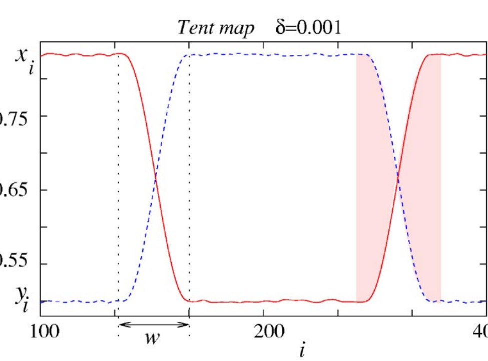

FIG. 4. Typical steady state profiles of the even (solid line) and the odd (dashed line) sublattices are shown for coupled tent maps with $\epsilon = 0.499$.

$$\alpha = \frac{m^2(2\delta+1) \pm (2\delta m - 1)}{m^2(2\delta+1) \pm m(4\delta+1)}.$$

The first solution corresponds to $z_{k+2}=z_k$. The kink is generated only when the other solution $z_{k+2}=\alpha$ is being chosen dynamically. Now, taking z_{k+1} and z_{k+2} as initial conditions, one can obtain z_{k+2+i} for $i \geqslant 1$ iteratively

$$\begin{pmatrix} z_{k+i+1} \\ z_{k+i+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -2\cos(\theta) \end{pmatrix}^i \begin{pmatrix} z_{k+1} \\ z_{k+2} \end{pmatrix}, \tag{8}$$

where $\theta = 2\tan^{-1}(\sqrt{\delta/\epsilon_B})$. For a site i far from k the solution of (8) can be approximated to linear order in δ as, $z_{k+2i} \approx \alpha_- + \Delta_i$ and $z_{k+2i+1} \approx \alpha_+ - \Delta_i$, where $\Delta_i = i^2\delta/2m(m+1)$. The width of the kink is thus $w=2n$, such that $\Delta_n = \alpha_+ - \alpha_-$. Clearly, w diverges as $1/\sqrt{\delta}$.

Discussion and conclusion. Let us first discuss the generalizations of the coupled PLMs of Eq. (6) to higher dimensions. A similar linear analysis would result in the same phase diagram as shown in Fig. 1. Our numerical simulations in two and three dimensions [18] verify that both, ST at ϵ_A and the desynchronization transition at ϵ_B are discontinuous.

The first-order STs found for the PLMs are quite general. Every other chaotic CMLs we studied in one dimension, for example $f(z)=4z(1-z)$, $\sin(\pi z)$, and $\sqrt{27}z(1-z^2)/2$ undergo a discontinuous transition from a multistable to a synchronized phase. Note that the linear stability boundary no longer represents the phase boundary of ST. In all these cases the synchronized phase loses stability for strong enough couplings; however, contrary to the coupled *tent maps* the desynchronization transitions are found to be continuous [18]. Interestingly for the power law map, $f(z)=1-|2z-1|^q$, one can even tune the nature of the desynchronization transition to be first or second order by tuning the parameter q. The transition turns out to be continuous for $q>1$, where the maps are differentiable. Details of these studies will be reported elsewhere.

In conclusion, we show that for a system of diffusively coupled chaotic maps, an effective delay introduced dynamically between sublattices can enforce synchronization by suppressing chaos. This synchronization transition which occurs as the system enters from a multistable region to a single "common fixed point" in phase space turns out to be discontinuous. For a single parameter family of coupled linear chaotic maps the phase boundary of the synchronized phase could be calculated exactly in any d dimension. Numerical studies of several other nonlinear maps suggests that the discontinuous synchronization transition is a generic feature of CMLs with *delay*. From the analogy between synchronization transition and nonequilibrium wetting process, previous studies [10] predicted a first-order phase transition within the DP regime where the largest Lyapunov exponent Λ is negative at the transition point. On the contrary, for the delay-induced synchronization discussed here, Λ changes sign exactly at the transition point. Since the synchronized phase in our model is not chaotic, corresponding synchronization transition cannot be modeled by Eq. (1), which assumes stochasticity. It would be of interest to study if Eq. (1) without the noise term can reproduce the phenomenology described here.

The *delay* brings in another interesting feature, namely, desynchronization, which occurs as the fixed point becomes unstable and then the even and odd sublattices fluctuate about two different common fixed points. Contrary to synchronization, which always occurs discontinuously, desynchronization can occur as a first- or second-order transition. Numerical study of several maps [18] suggests that the delay-induced desynchronization transition is continuous for maps which are differentiable everywhere.

We thank D. Mukamel and E. Levin for fruitful comments and discussions.

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